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On the equivalence theorem for S-matrix elements

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Abstract. We attempt to prove the equivalence theorem for S-matrix elements. We distinguish between the dynamical aspect—which ensures that the dynamics is coordinate-independent, and requires care in preservation of the order of noncommuting operators—and the kinematical aspect, which ensures that asymptotic fields are unchanged. It is the latter condition which causes difficulty; whilst trivial for the free field we give nontrivial examples which violate it.

1. Introduction

A great deal has recently been made of the equality of S-matrix elements from Lagrangians which are formally related by point transformations of the fields (Chisholm 1961, Kamefuchi *et al.* 1961). In particular this has been used in discussions of the different Lagrangians giving nonlinear realizations of chiral symmetry (Coleman *et al.* 1969, Weinberg 1968); a coordinate-independent formulation of the chiral symmetry group has been developed by Barnes and Isham (1970) by means of a geometric approach. This has proved of importance in discussing the large momentum transfer behaviour for nucleon electromagnetic form factors (Martin and Taylor 1970), and in the formulation of non-abelian gauge theories (Salam and Strathdee 1970). It is evidently desirable to have a proof that S-matrix elements are indeed invariant under point transformations of the fields. This paper is devoted to a discussion of this problem.

The original statement and proof of this invariance, the equivalence theorem, was given by Chisholm (1961). This was amended and extended by Kamefuchi *et al.* (1961), who made extensive use of certain asymptotic conditions for powers of an interacting field. Whilst these were weaker than those used by Chisholm, they still contained assumptions which interactions might not satisfy. An alternative proof by field theoretic functional methods has been recently suggested (Salam and Strathdee 1970); this apparently avoids the problematical conditions of the earlier discussions. However, it is apparent that such an approach is applicable to non-relativistic quantum mechanical problems; in this situation we will see later that this avoidance is illusory.

Before we go into detail about this situation, which we do in § 3, we differentiate in the next section between the dynamical and kinematical aspects of the problem. The former, trivial in principle, is that the dynamical development is independent of the coordinates used to describe it. However, we find that this independence is not so trivial to preserve in practice, partly owing to problems of ordering of canonically conjugate operators. The kinematical aspect is that of ensuring that the asymptotic fields are preserved under coordinate transformations. We find that this is so if a natural normalization condition is satisfied; this condition is different from that occurring in previous proofs (Chisholm 1961, Kamefuchi *et al.* 1961, Salam and Strathdee 1970). After the discussion promised in § 3 we consider theories obtained by point transformations of a free field in § 4; certain of these are shown to satisfy the above-mentioned normalization condition. In § 5 we show that the original form of the equivalence theorem is indeed valid in the tree graph approximation. In the final section we discuss the relevance of the theorem for chirally symmetric Lagrangians, and those with symmetry breaking given by the PCAC hypothesis (Chang and Gursey 1967).

2. Dynamics and kinematics

We consider in this section the theory of a spinless neutral meson, described by a field $\psi(x)$; we are interested in coordinate transformations $\psi(x) \to \phi(x) = \phi\{\psi(x)\}$. The equivalence theorem (Chisholm 1961, Kamefuchi *et al.* 1961) states that for a large class of suitable functions ϕ the S-matrix elements for processes in which the meson is described by the field coordinate ϕ are the same as those in terms of the old coordinates ψ. We recognize that there are two distinct parts to this theorem. One is concerned with the dynamics, and states that the time development of a system of such mesons is independent of the coordinates used to describe it. In other words, the Green functions $\langle 0|T[\phi(x_1) \ldots \phi(x_n)]|0\rangle$, where $|0\rangle$ is the vacuum state and T the usual time ordering symbol, can be calculated from a Hamiltonian theory in which the Hamiltonian is given either in terms of the field ϕ or the field ψ; the result is independent of which choice is made. This point is not trivial to ensure in practice owing to the problems of ordering of products of field operators and their time derivatives. This is especially so for interactions involving field derivatives, leading to apparently noncovariant expressions. For derivative coupling of vector mesons Matthews (1949) showed that covariance is obtained by a modified time ordering in which derivatives of fields are to be taken after the time ordering. However, this result may not apply to the more general situation, when high powers of time derivatives of the field may occur in the interaction; one of us (B.W.K.) has shown that this is still true in certain cases in low orders of perturbation theory. We will not discuss this further here, since we are mainly concerned with principle in this paper.

The kinematic aspect is that of ensuring that the asymptotic fields are unchanged under the coordinate transformation. In other words, we require

$$\lim_{x^0 \to \mp\infty} [\phi\{\psi(x)\} - \psi_{\substack{\text{in}\\ \text{out}}}(x)] = 0 \tag{1}$$

where the asymptotic limit is weak. This condition can be derived from

$$(2p_0)^{1/2} \langle 0|\phi\{\psi(x)\}|1, \boldsymbol{p}\rangle = \mathrm{e}^{-ipx} \tag{2}$$

where $|1, \boldsymbol{p}\rangle$ is a one-particle state of momentum $\boldsymbol{p}$, and energy $p^0 = (m^2 + \boldsymbol{p}^2)^{1/2}$, m being the particle mass. Condition (2) is one of the basic ones in the Haag–Ruelle proof (see for example Jost 1965) of the asymptotic condition (1). We prefer that (1) be satisfied directly, but we are prepared to settle for (2). What are the conditions on ϕ in order that either of these conditions be true? In the earlier work of Kamefuchi *et al.* (1961) it was claimed that (1) was true if

$$\phi'\{0\} = 1. \tag{3}$$

The justification for this can easily be given if it is assumed that

$$\lim_{x^0 \to \mp\infty} [\phi\{\psi(x)\} - \phi\{\psi_{\substack{\text{in}\\ \text{out}}}(x)\}] = 0 \tag{4}$$

for then all the terms in $\phi\{\psi_{\substack{\text{in}\\ \text{out}}}(x)\}$ will vanish asymptotically, except for $\phi'(0)\psi_{\substack{\text{in}\\ \text{out}}}(x)$. Equation (1) is then satisfied if (3) is true. However, (4) is not necessarily true. To see this we may rewrite the left hand side of (2) at $x = 0$, using the assumed asymptotic condition for ψ, as

$$A = \mathrm{i} \int \mathrm{e}^{-\mathrm{i}py} K_y \langle 0 | T[\phi\{\psi(0)\}\psi(y)] | 0 \rangle \, \mathrm{d}^4 y \Big|_{p^2 = m^2} \tag{5}$$

where $K_y = (\Box_y{}^2 + m^2)$. If we assume that ϕ has a Taylor series expansion in its variable

$$\phi\{Z\} = \sum_{n \geqslant 0} a_n Z^n$$

then

$$A = \mathrm{i} \sum_{n \geqslant 0} a_n \int \mathrm{e}^{-\mathrm{i}py} K_y \langle 0 | T[\psi^n(0)\psi(y)] | 0 \rangle \, \mathrm{d}^4 y \Big|_{p^2 = m^2} \tag{6}$$

which graphically has the form of the sum of contributions of self energy type as shown in figure 1. We do not expect that the coefficient of a_n, $n > 1$, in A be zero;

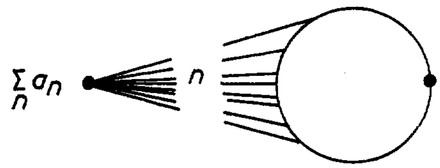

Figure 1. The sum of terms of self-energy character which are equal to the normalization constant A of equation (5).

in the simple case that ψ has a quartic self interaction $\lambda\psi^4$ each coefficient will have a power series expansion in powers of λ with definitely nonzero coefficients. Since A, defined by (5), would be equal to $\langle 0 | \phi\{\psi_{\text{in}}(0)\} | 1 \rangle$ if (4) were valid, and so equal to $\phi'(0)$, then we see that a contradiction would arise in such a case. In other words (4) cannot be valid; its breakdown is due to the single particle states created by the higher powers of ψ in $\phi(\psi)$ and which do not enter in $\phi(\psi_n)$. We note that the coefficient of a_1 in A is unity if the asymptotic condition is satisfied by ψ.

Returning to (2) at $x = 0$, we see that the validity of the asymptotic condition for ϕ, from the definition (5), is

$$A = 1. \tag{7}$$

It may be difficult to obtain the asymptotic condition (1) from (2) or equivalently (7), in the case of $\phi(x)$ being a polynomial in ψ at x, owing to the difficulty of defining powers of $\psi(x)$. However, we expect that an argument on the level of perturbation theory (in the interaction picture for ψ) will allow (1) to be derived from (7).

It is natural to ask why it is that the earlier condition (3) is not valid, and is replaced by (7). The reason for this can be seen in the breakdown of (4): there are

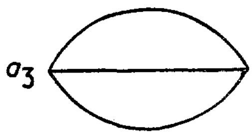

Figure 2. A nonzero contribution in perturbation theory to the normalization constant A for the case of a quartic self-interaction.

single particle contributions in $\phi\{\psi\}$ from the higher powers of ψ in the Taylor series expansion. Indeed we have given the general form of these contributions in figure 1, in terms of Green functions for ψ. A simple contribution is shown in figure 2, in the case of the self interaction of ψ possessing a quartic term as we described earlier. This and similar contributions do not vanish, as was remarked before, and imply that the earlier discussions (Chisholm 1961, Kamefuchi *et al.* 1961) have to be altered by replacing (3) with (7).

3. Functional integration

The recent discussion of the equivalence theorem (Salam and Strathdee 1970) appears to avoid the difficulty related to (3) described in the preceding section. We will briefly describe the techniques of functional integration used in reference (5) to show precisely how the dynamical and kinematical parts of the equivalence theorem arise there. The Green functions for the field ϕ have the generating functional

$$Z_\phi(J) = Z_\phi{}^{-1}\prod_x \int \mathrm{d}\pi_{(x)}\,\mathrm{d}\phi_{(x)} \exp\left[\mathrm{i}\int \mathrm{d}^4x\{\pi\dot{\phi}-H^1+J\phi\}\right]$$

where $H^1 = H^1(\pi, \phi)$ is the Hamiltonian density for the field ϕ with canonical momentum π and $\mathrm{d}\phi_{(x)}$, $\mathrm{d}\pi_{(x)}$ are the functional integral measures whose nature we make no attempt to specify precisely, and Z_ϕ is a constant such that $Z_\phi(0) = 1$. Under the contact transformation $(\chi, \psi) \leftrightarrow (\pi, \phi)$ we may rewrite

$$Z_\phi(J) = Z_\phi{}^{-1}\prod_x \int \mathrm{d}\chi_{(x)}\,\mathrm{d}\psi_{(x)} \exp\left[\mathrm{i}\int \mathrm{d}^4x\{\chi\dot{\psi}-H+J\phi\}\right]$$

where $H = H(\chi, \psi)$ is the new Hamiltonian, the functional Jacobian of the transformation being unity. Since evidently $Z_\psi = Z_\phi$ where

$$Z_\psi(J) = Z_\psi{}^{-1}\prod_x \int \mathrm{d}\chi_{(x)}\,\mathrm{d}\psi_{(x)} \exp\left[\mathrm{i}\int \mathrm{d}^4x\,\{\chi\dot{\psi}-H+J\psi\}\right]$$

then

$$Z_\phi(J) = Z_\psi{}^{-1}\prod_x \int \mathrm{d}\chi_{(x)}\,\mathrm{d}\psi_{(x)} \exp\left[\mathrm{i}\int \mathrm{d}^4x\{\chi\dot{\psi}-H+J\phi\}\right]. \tag{8}$$

But this is just the statement of the dynamical aspect of the equivalence theorem; it enables the Green functions for the field ϕ to be evaluated using the dynamics expressed in terms of ψ. The kinematical aspect of the equivalence theorem emerges when it is attempted to use (8) to calculate the S-matrix elements for the field ϕ in terms of those for ψ. The Green functions $G_\phi(x_1 \dots x_n)$ for ϕ are defined as

$$G_\phi(x_1 \dots x_n) = \left.\frac{\delta^n Z_\phi(J)}{\delta J(x_1)\dots\delta J(x_n)}\right|_{J=0}$$

so are related to those for ψ, from (8), by

$$G_\phi(x_1 \dots x_n) = \prod_i \phi\left\{\frac{\delta}{\delta J(x_i)}\right\} Z_\psi(J)\Big|_{J=0} \tag{9}$$

where we have assumed that $\psi \to \phi$ is a point transformation. If we assume an asymptotic condition for the field ϕ as well as for ψ we may obtain, from (9) and the

Taylor expansion of ϕ, the on-mass-shell S-matrix elements for ϕ, without the no-scattering terms, using the standard reduction procedure

$$\begin{aligned} S_\phi(p_1 \dots p_n) &= \prod_i (-p_i^2+m^2) \int \mathrm{d}x_i\, \mathrm{e}^{\mathrm{i}p_i x_i}\, G_\phi(x_1 \dots x_n) \\ &= \prod_i (-p_i^2+m^2) \sum_{\{n_i\}} \prod_i a_{n_i} \prod_j \int \mathrm{d}x_j\, \mathrm{e}^{\mathrm{i}p_j x_j} \\ &\quad \times G_\psi(\underbrace{x_1 \dots x_1}_{n_1}, \dots \underbrace{x_n \dots x_n}_{n_n})\Big|_{p_i^2=m^2}. \end{aligned} \tag{10}$$

The pole contributions on the right hand side of (10) can be written in terms of the S-matrix elements for ψ as $A^n S_\psi(p_1 \dots p_n)$. We see at this point that the spectrum of the two theories must coincide; this is certainly to be expected. What is more, if the S-matrix elements for ψ satisfy the usual unitarity conditions then it will not be possible for the S-matrix elements for ϕ to satisfy the same conditions. In other words, in order that the S-matrix elements for ϕ also be unitary we have to require $A = 1$; this is the first condition (7). Of course, if S is identically one, the 'free field' case, then any value of A will be satisfactory. This case is trivial, because all the on-mass-shell S-matrix elements given by (10) are identically zero. We see then that the kinematical aspect of the equivalence theorem remains unchanged by using functional integral methods.

What about the dynamical aspects of the theorem? It is here that the functional integral formulation of quantum field theory (and of quantum mechanics also) hides the real problem. We can see this in particular for the quantum mechanical motion of a particle in one dimension. The ambiguity of ordering arising in the case of a Hamiltonian such as $\frac{1}{2}q^2p^2$ can readily be seen to arise from different ways of calculating approximants to the integrands of the functional integrals, in particular approximants to $\dot{q}$. This means that the problem of translating the dynamics from one set of coordinates to another in the Hamiltonian formalism is no less trivial in the functional integral formalism than it was in the operator formalism.

4. The free field

We will discuss this as a model for which the transformations satisfying the condition (7) can be determined (although condition (2) is no longer necessary, as was remarked in the last section). Using only the canonical commutation relations for the field ψ,

$$A = \mathrm{i} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \langle 0 | T[\phi\{\psi(0)\} \mathscr{J}(x)] | 0 \rangle |_{p^2=m^2} + \langle 0 | \phi'\{\psi(0)\} | 0 \rangle \tag{11}$$

where $\mathscr{J} = (\Box + m^2)\psi$. For the free field case $\mathscr{J} = 0$ so

$$A = \langle 0 | \phi'\{\psi(0)\} | 0 \rangle$$

if we take ϕ to be normally ordered (in ψ) then ϕ' defined by

$$\delta^3(x)\, \phi'\{\psi(0)\} = \mathrm{i}[(\dot{\psi} x), \phi\{\psi(0)\}]$$

is also normally ordered (in ψ). Hence $A = \phi'\{0\}$, and condition (7) reduces to the original condition (3).

5. Equivalence theorem for tree graphs

The tree graph approximation is the only one for which numerical calculations have been made from chiral Lagrangians. It is possible to show that for this approximation the original form of the equivalence theorem is valid.

We use the argument due to Lee and Nieh (1968). Define a new Lagrangian

$$\mathscr{L}_c(\psi, \partial\psi) = \frac{1}{c^2}\mathscr{L}(c\psi, c\partial\psi)$$

where $\mathscr{L}(\psi, \partial\psi)$ is the original Lagrangian, and a new transformation

$$\psi \to \phi_c = f_c(\psi) = \frac{1}{c}f(c\psi).$$

Let $\mathscr{M}(\phi, \partial\phi) \equiv \mathscr{L}(\psi, \partial\psi)$ be the Lagrangian written in terms of the transformed field $\phi = f(\psi)$. Then the Lagrangian $\mathscr{M}_c$ for the transformed field ϕ_c will be given by

$$\mathscr{M}_c(\phi_c, \partial\phi_c) = \mathscr{L}_c(\psi, \partial\psi)$$

which is in fact equal to

$$\frac{1}{c^2}\mathscr{M}(c\phi_c, c\partial\phi_c).$$

We now assume that

$$\langle 0|T\left[\prod_{i=1}^{n} f_c\{\psi(x_i)\}\right]|0\rangle = \langle 0|T\left[\prod_{i=1}^{n}\phi_c(x_i)\right]|0\rangle \tag{12}$$

where the left hand side of (12) is calculated from $\mathscr{L}_c$, the right hand side from $\mathscr{M}_c$. Provided the order of noncommuting terms going from $\mathscr{L}_c$ to $\mathscr{M}_c$ is preserved, this will be true. Now the tree graph contribution to both sides of (12) is the term of lowest order in c (Lee and Nieh 1968), so they must be equal. In the evaluation of mass-shell S-matrix elements, we see from figure 1 that the only term of A that is retained in the tree approximation is $n = 1$, that is, in the tree approximation $A = a_1$, so the condition $A = 1$ becomes $a_1 = 1$, which is the original form of the equivalence theorem.

6. Equivalence for chiral theories

As remarked in the introduction there has been a great deal of use of the equivalence theorem in theories which are chirally invariant or in which this is broken to give PCAC. Of course originally this involved only equivalence for tree graphs, but there has been great interest recently in true nonpolynomial Lagrangian theories (Proceedings of the 15th High Energy Physics Conference, Kiev 1970). We have not been able to prove it except for the tree graph approximation. However, the equivalence theorem is not relevant when PCAC is imposed; the choice of coordinates for the chirally invariant part of the Lagrangian determines the symmetry breaking term but not in a coordinate-invariant fashion, that is, the new Lagrangian, still satisfying PCAC, is not obtained from the original one by a coordinate transformation. This can be seen for example by explicit analysis (Chang and Gursey 1967); it is also apparent from the inequality of the π–π scattering lengths calculated in different coordinate frames (Chang and Gursey 1967). However, in the zero mass pion limit these scattering lengths become coordinate-independent again, in the tree approximation, as is to be expected from the discussion of § 5.

Whilst it would be preferable to have a general proof of the applicability, or not, of the theorem to the commonly used transformations, it is evident that the inequivalent results when PCAC is satisfied reduce its physical importance. Indeed, it would appear that nature has somehow chosen a particular coordinate frame in which to perform its mysteries. This raises the questions as to what reasons can guide us towards choosing this frame and what it actually is. An interesting suggestion has recently been made by Charap (1970) that there is a unique coordinate frame in which the divergences of chiral invariant meson theory cancel, and the self mass of the pion is zero. It would certainly be convenient if nature had chosen this coordinate frame since not only would finite results be obtained but also a unique form for the symmetry-breaking PCAC terms and so for π–π scattering lengths. It may well be that such cancellation is the only way to obtain finite results for chirally invariant meson theories. We hope to return to this elsewhere.

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